

Notes on normed algebras, 2

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Let \mathcal{A} be a finite-dimensional commutative algebra over the complex numbers, with identity element e . Thus \mathcal{A} is a finite-dimensional complex vector space equipped with an additional binary operation of multiplication which satisfies the usual rules of associativity, commutativity, and distributivity, and e is a nonzero element of \mathcal{A} such that $e x = x$ for all $x \in \mathcal{A}$. As a basic class of examples, one can take X to be a finite set, and \mathcal{A} the algebra of complex-valued functions on X , with respect to pointwise multiplication.

As another class of examples, suppose that A is a commutative semigroup with identity. In other words, A is a set equipped with an identity element 0 and a binary operation $+$ which satisfies the usual associativity and commutativity rules, and with $x + 0 = x$ for all $x \in A$. Let \mathcal{A} be the vector space of complex-valued functions on A , and define the convolution of two such functions f_1, f_2 by

$$(1) \quad (f_1 * f_2)(z) = \sum_{x+y=z} f_1(x) f_2(y).$$

More precisely, the sum is taken over all $x, y \in A$ such that $x + y = z$. One can check that this operation of convolution satisfies the commutative and associative laws, and of course it is linear in f_1, f_2 .

For each $x \in A$, let δ_x denote the function on A which is equal to 1 at x and to 0 at all other elements of A . The semigroup operation on A corresponds exactly to convolution of these functions. The function δ_0 serves as an identity element for the operation of convolution. Thus the vector space of complex-valued functions on A becomes a commutative algebra with respect to convolution.

Let \mathcal{A} be a finite-dimensional commutative algebra over the complex numbers. By an ideal in \mathcal{A} we mean a linear subspace \mathcal{I} of \mathcal{A} such that if $x \in \mathcal{I}$

and $a \in \mathcal{A}$, then $ax \in \mathcal{I}$ as well. We say that an ideal \mathcal{I} in \mathcal{A} is proper if it is not all of \mathcal{A} . As a special case, if $x \in \mathcal{A}$, then we get an ideal $\mathcal{I}(x)$ consisting of ax as a runs through all elements of \mathcal{A} . This ideal is proper if and only if x does not have an inverse in \mathcal{A} .

Suppose that \mathcal{I} is a proper ideal in the commutative finite-dimensional algebra \mathcal{A} over the complex numbers. Thus \mathcal{I} is a linear subspace of \mathcal{A} , and we can form the quotient space \mathcal{A}/\mathcal{I} initially as a vector space of positive dimension equal to the dimension of \mathcal{A} minus the dimension of \mathcal{I} . There is an associated quotient mapping from \mathcal{A} onto \mathcal{A}/\mathcal{I} , which sends an element of \mathcal{A} to the corresponding element of the quotient space. The kernel of this mapping is exactly equal to \mathcal{I} .

By the usual arguments, the operation of multiplication on \mathcal{A} leads to a similar operation on the quotient \mathcal{A}/\mathcal{I} , because \mathcal{I} is an ideal. This operation on the quotient \mathcal{A}/\mathcal{I} is commutative, because the initial operation on \mathcal{A} is commutative. The image of the identity element e in the quotient is an identity element in the quotient, and it is nonzero because the ideal \mathcal{I} is proper, and therefore does not contain e . In short the quotient \mathcal{A}/\mathcal{I} is itself a finite-dimensional commutative algebra.

A maximal ideal \mathcal{I} in \mathcal{A} is a proper ideal with the property that any ideal in \mathcal{A} which contains \mathcal{I} is either equal to \mathcal{I} or to \mathcal{A} . Every proper ideal in \mathcal{A} is contained in a maximal ideal, namely, in a proper ideal of maximal dimension. The only proper ideal in \mathcal{A} is the ideal containing only 0 if and only if \mathcal{A} is a field, which is to say that every nonzero element of \mathcal{A} has an inverse in \mathcal{A} . A finite-dimensional algebra over the complex numbers which is a field is equal to the span of its identity element, which is to say that it has dimension 1 and is isomorphic to the complex numbers. This follows from the fundamental theorem of algebra.

Suppose that \mathcal{A} is a finite-dimensional commutative algebra over the complex numbers with identity element e , and that \mathcal{I} is a proper ideal in \mathcal{A} . Thus we get a nonzero homomorphism from \mathcal{A} onto the quotient \mathcal{A}/\mathcal{I} . If \mathcal{I} is a maximal ideal, then \mathcal{A}/\mathcal{I} is one-dimensional and isomorphic to the complex numbers. Equivalently, we get a homomorphism from \mathcal{A} onto the complex numbers whose kernel is equal to \mathcal{I} and which sends the identity element e in \mathcal{A} to the complex number 1. Conversely, a homomorphism from \mathcal{A} onto the complex numbers has kernel equal to a maximal ideal in \mathcal{A} and sends e to 1.

Suppose that \mathcal{A} is a finite-dimensional commutative algebra over the complex numbers with identity element e , and that ϕ is a homomorphism from

\mathcal{A} onto the complex numbers, which then takes e to 1 automatically. If x is an invertible element of \mathcal{A} , then $\phi(x)$ is an invertible complex number, which means that $\phi(x) \neq 0$. Thus $\phi(x) = 0$ implies that x does not have an inverse in \mathcal{A} . If we start with $x \in \mathcal{A}$ which is not invertible, then the ideal $\mathcal{I}(x)$ generated by x is proper, and is therefore contained in a maximal ideal. It follows that there is a homomorphism ϕ from \mathcal{A} onto the complex numbers such that $\phi(x) = 0$.

Recall that the spectrum of an element x of \mathcal{A} is defined to be the set of complex numbers λ such that $\lambda e - x$ does not have an inverse in \mathcal{A} . The preceding remarks imply that the spectrum is the same as the set of complex numbers which arise as $\phi(x)$ for some homomorphism ϕ from \mathcal{A} onto the complex numbers. We also know that the spectrum of x is a nonempty finite set of complex numbers.

Now suppose that \mathcal{A} is a finite-dimensional commutative algebra over the complex numbers equipped with a norm $\|\cdot\|$ such that $(\mathcal{A}, \|\cdot\|)$ is a normed algebra. Thus $\|\cdot\|$ is a nonnegative real-valued function on \mathcal{A} such that $\|x\| = 0$ if and only if $x = 0$, $\|\alpha x\| = |\alpha| \|x\|$ for all complex numbers α and $x \in \mathcal{A}$, $\|e\| = 1$, and the norm of a sum or product of two elements of \mathcal{A} is less than or equal to the corresponding product of the norms. If λ is a complex number and x is an element of \mathcal{A} such that $\|x\| < |\lambda|$, then $\lambda e - x$ is invertible in \mathcal{A} , because one can sum the series $\sum_{j=0}^{\infty} \lambda^{-j} x^j$. In other words, $|\lambda| \leq \|x\|$ for all λ in the spectrum of x . If ϕ is a homomorphism from \mathcal{A} onto the complex numbers, then $|\phi(x)| \leq \|x\|$ for all $x \in \mathcal{A}$, since $\phi(x)$ is in the spectrum of x .

Let X be a finite nonempty set, and let \mathcal{A} be the algebra of complex-valued functions on X , with respect to pointwise addition and multiplication. If \mathcal{I} is an ideal in \mathcal{A} , then one can show that there is a subset E of X such that \mathcal{I} consists exactly of the functions f on X which satisfy $f(p) = 0$ when $p \in E$. Moreover, \mathcal{I} is a proper ideal if and only if there is a point $p \in X$ such that \mathcal{I} consists exactly of the functions f on X such that $f(p) = 0$. Thus the homomorphisms from \mathcal{A} onto the complex numbers are exactly of the form $f \mapsto f(p)$ for $p \in X$.

There is a natural norm in this setting, defined by saying that the norm of a function f on X is equal to the maximum of $|f(p)|$ over $p \in X$. The spectrum of a function f on X , as an element of the algebra of functions on X , is equal to the set of values of f . Thus the norm of f is equal to the maximum of the absolute values of the complex numbers in the spectrum of f .

Let A be a finite commutative semigroup, and let \mathcal{A} be the algebra of complex-valued functions on A with respect to convolution as described earlier. Suppose that ϕ is a homomorphism from \mathcal{A} onto the complex numbers. This leads to a mapping Φ from A to the complex numbers, defined by $\Phi(a) = \phi(\delta_a)$ for $a \in A$, where δ_a is the function on A which is equal to 1 at a and to 0 at other elements of A . When $a = 0$, this is the identity element of \mathcal{A} . The requirement that ϕ map \mathcal{A} onto the complex numbers is equivalent to ϕ mapping δ_0 to 1, which is the same as $\Phi(0) = 1$.

Because $\phi : \mathcal{A} \rightarrow \mathbf{C}$ is a homomorphism, Φ should be a homomorphism from A into \mathbf{C} as a semigroup with respect to multiplication. In other words we should have $\Phi(a + b) = \Phi(a) \Phi(b)$ for all $a, b \in A$. Under the assumption that ϕ is linear, this condition on Φ is equivalent to $\phi(f_1 * f_2) = \phi(f_1) \phi(f_2)$ for all $f_1, f_2 \in \mathcal{A}$. This follows from the fact that \mathcal{A} is spanned by the functions δ_a , $a \in A$.

Let us define a norm on \mathcal{A} in this situation by setting $\|f\|$ equal to $\sum_{a \in A} |f(a)|$ for all complex-valued functions on A . One can check that

$$(2) \quad \|f_1 * f_2\| \leq \|f_1\| \|f_2\|$$

for all complex-valued functions f_1, f_2 on A in this situation. Also, $\|\delta_a\| = 1$ for all $a \in A$, and in particular the norm of the identity element of \mathcal{A} is equal to 1.

Suppose that ϕ is a homomorphism from \mathcal{A} onto the complex numbers, and let Φ be the corresponding homomorphism from A into the multiplicative semigroup of complex numbers. The image of A under Φ is a finite subsemigroup of the multiplicative semigroup of complex numbers, and therefore $|\Phi(a)| \leq 1$ for all $a \in A$. Indeed, if we choose $a \in A$ so that $|\Phi(a)|$ is maximal, then $|\Phi(a + a)|$ is equal to $|\Phi(a)|^2$, and it is less than or equal to $|\Phi(a)|$ by maximality. Thus $|\Phi(a)| \leq 1$, as desired. Of course $\Phi(0) = 1$, so that the maximum of $|\Phi|$ on A is equal to 1. If f is a complex-valued function on A , then we can use linearity of ϕ to write $\phi(f)$ explicitly as $\sum_{a \in A} \Phi(a) f(a)$. We can apply absolute values to this sum and use the triangle inequality to obtain directly that $|\phi(f)| \leq \|f\|$.

References

- [1] W. Rudin, *Functional Analysis*, second edition, McGraw-Hill, 1991.